Distribution of the Scaled Condition Number of Single-spiked Complex Wishart Matrices

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(*Joint work with Prathapasinghe Dharmawansa and Yang Chen*)

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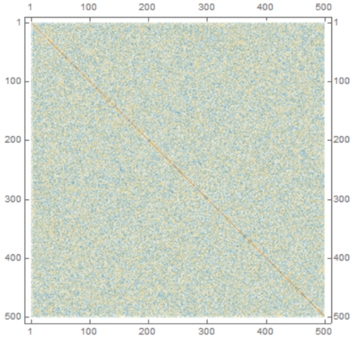
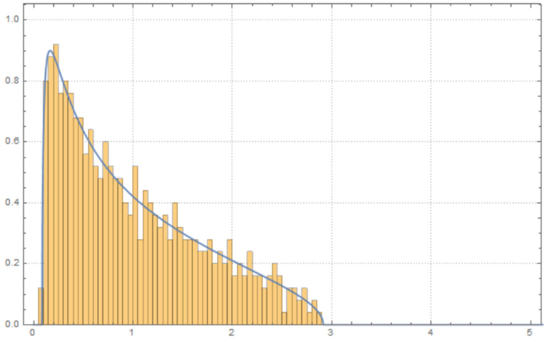
28$^{\text{th}}$ July, 2022

Why RMT?





Covariance: $\mathbf{\Sigma} = \mathbf{I} + \sum_{k=1}^{r} \eta_k \mathbf{u}_k \mathbf{u}_k^*, \; ||\mathbf{u}_k|| = 1$

Our Focus

$$\mathbf{X} \in \mathbb{C}^{n\times m} (m \geq n) \qquad \mathbf{X} = \begin{pmatrix} | & & | \\ \mathbf{x}_1 & \ldots & \mathbf{x}_m \\ | & & | \end{pmatrix}_{n\times m} \qquad \mathbf{x}_j \in \mathbb{C}^{n\times 1} \text{ for } j = 1, \ldots, m$$

$\mathbb{E}\{\mathbf{X}\} = \mathbf{O}$ $\quad$ $\mathbb{E}\left\{\mathbf{x}_j \mathbf{x}_j^*\right\} = \mathbf{\Sigma}$ $\quad$ Independent samples

$$\mathbf{X}_{n\times m} \sim \mathcal{CN}_{n,m}(\mathbf{O}, \mathbf{\Sigma} \otimes \mathbf{I}_m)$$

$$\mathbf{\Sigma} = \mathbf{I} + \sum_{k=1}^{r} \eta_k \mathbf{u}_k \mathbf{u}_k^* \quad \text{where } ||\mathbf{u}_k|| = 1$$

Spiked covariance (Johnstone, 2001)

- $r = 1$: $\mathbf{\Sigma} = \mathbf{I}_n + \eta \mathbf{u}\mathbf{u}^*$ Single-spiked/rank-one perturbation
- Signal detection, PCA, Factor models, Equal correlation MIMO model

Our Focus: Scaled Condition Number (Demmel Condition Number)

Wishart-Laguerre ensemble: $\mathbf{W} = \mathbf{X}\mathbf{X}^* = \sum_{j=1}^{m} \mathbf{x}_j \mathbf{x}_j^*$ ($\mathbf{W}$ Positive Definite for $m \geq n$)

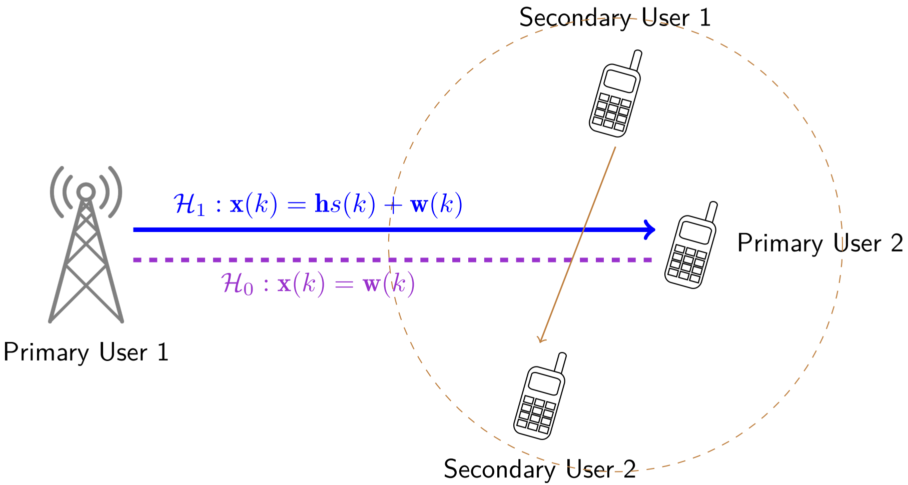
Eigen-decomposition: $\mathbf{W} = \mathbf{U}\text{Diag}(\lambda_1, \ldots, \lambda_n)\mathbf{U}^*$ with $0 < \lambda_1 \leq \ldots \leq \lambda_n$

$$\kappa_{\mathsf{SC}}^2(\mathbf{X}) = ||\mathbf{X}||_F^2 ||\mathbf{X}^\dagger||_2^2 = \frac{\sum_{i=1}^{n} \lambda_i}{\lambda_1}$$

- Origin: To measure degree of difficulty associated with numerical analysis problems
- Statistical characterization motivated by Probabilistic Analysis (Demmel, 1988, Spielman and Teng, 2002 etc.)

Note: Eigenvalues of $\boldsymbol{\Sigma} : 1 + \eta, \underbrace{1, \ldots, 1}_{n-1 \text{ times}} \implies \kappa_{\mathsf{SC}}^2(\mathbf{X}_{\boldsymbol{\Sigma}}) = n + \eta$

$\kappa^2_{\mathsf{SC}}(\mathbf{X})$ in Cognitive Radio Spectrum Sensing



- CR introduced by (Mitola and Maguire, 1999)
- A promising technology for 5G
- Already standardized: IEEE 802.22, 802.11af

$\kappa^2_{\mathsf{SC}}(\mathbf{X})$ in CR Spectrum Sensing: *Binary Hypothesis Testing*

$$\mathcal{H}_0 : \mathbf{x}(k) = \mathbf{w}(k), k = 1, \ldots, m$$

$$\mathcal{H}_1 : \mathbf{x}(k) = \mathbf{h}s(k) + \mathbf{w}(k), k = 1, \ldots, m$$

Population covariance: $\mathbf{R} = \mathbb{E}\{\mathbf{x}(k)\mathbf{x}(k)^*\} = \begin{cases} \sigma^2 \mathbf{I}_n & \text{under } \mathcal{H}_0 \\ \sigma^2 \mathbf{I}_n + \gamma \mathbf{h}\mathbf{h}^* & \text{under } \mathcal{H}_1 \end{cases}$

Sample covariance: $\hat{\mathbf{R}} = \frac{1}{m}\sum_{j=1}^{m} \mathbf{x}(j)\mathbf{x}(j)^* = \dfrac{1}{m}\mathbf{X}\mathbf{X}^*$: $\mathbf{X} = [\mathbf{x}(1) \ \ldots \ \mathbf{x}(m)]_{n \times m}$

Instead consider $\mathbf{W} = \mathbf{X}\mathbf{X}^* \leftarrow$ *Scaling won't affect* $\kappa^2_{\mathsf{SC}}(\mathbf{X})$

Test Statistic: $\kappa^2_{\mathsf{SC}}(\mathbf{X}) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \xi$

(Zeng & Liang, 2009, Axell *et al.*, 2012)

- False alarm rate - p.d.f. under $\mathcal{H}_0$ (Zhong *et. al.*, 2011)
- Detection power - p.d.f under $\mathcal{H}_1$??

$\kappa^2_{\mathsf{SC}}(\mathbf{X})$ for Adaptive MIMO Transmission Characterization

MIMO: Trade-off between *Rate* and *Reliability*

Diversity
send same symbol
through multiple paths
to increase reliability

Multiplexing
send different symbols
through different paths
to increase data rate

MIMO model: $\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n} \rightarrow$ Under ML detection: $\Pr(\epsilon|\mathbf{H}) = f(d^2_{\text{min-rx}})$

Antenna correlation (Correlated Rayleigh): $\mathbf{H} \sim \mathcal{CN}_{n,m}(\mathbf{0}, \boldsymbol{\Sigma} \otimes \mathbf{I}_m)$

Instantaneous decision: $\kappa^2_{\mathsf{SC}}(\mathbf{H}) < \frac{d^2_{\text{min-tx,multiplex.}}}{d^2_{\text{min-tx,diversity}}} \implies$ Use multiplexing

Statistical properties??

(Heath and Paulraj, 2005)

And More...

MATHEMATICS OF COMPUTATION
VOLUME 50, NUMBER 182
APRIL 1988, PAGES 449–480

The Probability That A Numerical Analysis Problem Is Difficult

By James W. Demmel

Abstract. Numerous problems in numerical analysis, including matrix inversion, eigenvalue calculations and polynomial zerofinding, share the following property: The difficulty of solving a given problem is large when the distance from that problem to the

JOURNAL OF MULTIVARIATE ANALYSIS 4, 265–282 (1974)

On the Evaluation of Some Distributions that Arise in Simultaneous Tests for the Equality of the Latent Roots of the Covariance Matrix

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In this paper, the authors consider the evaluation of the distribution functions of the ratios of the intermediate roots to the trace of the real Wishart matrix

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **50** (2017) 345201 (23pp) https://doi.org/10.1088/1751-8121/aa7d0e

Smallest eigenvalue density for regular or fixed-trace complex Wishart–Laguerre ensemble and entanglement in coupled kicked tops

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Department of Physics, Shiv Nadar University, Gautam Buddha Nagar, Uttar Pradesh—201314, India

What We Know

"On the distribution of a scaled condition number"
A. Edelman
Math. Comp., 1992.

Corollary 3.1. *Let* $h_n(x)$ $(x \geq \sqrt{n})$ *be the density of the condition number* $\kappa_D(A)$ *for complex matrices. Then*

$$h_n(x) = 2n(n^2 - 1)x^{1-2n^2}(x^2 - n)^{n^2-2}.$$

Corollary 3.2. *The probability distribution of* κ_D *is given in the complex case by*

$$P(\kappa_D \geq x) = 1 - (1 - n/x^2)^{n^2-1}, \qquad x > \sqrt{n}.$$

The above result allows us to verify that indeed

Corollary 3.3. *For fixed* n, *as* $x \to \infty$,

$$P(\kappa_D \geq x) \sim n(n^2 - 1)/x^2.$$

Figure 2: For square $\mathbf{X} \sim \mathcal{CN}_{n,n}(\mathbf{0}, \mathbf{I}_n \otimes \mathbf{I}_n)$

What We Know

"Distribution of the Demmel condition number of Wishart matrices"
C. Zhong et. al.
IEEE Trans. Commun., 2011.

Theorem 1: For a complex central Wishart matrix $\mathbf{W} \sim \mathcal{CW}_n(m, \mathbf{I})$, the p.d.f. of the Demmel condition number can be expressed as

$$p(x) = \frac{K_{n,m}\Gamma(nm)}{\Gamma(n)x^{nm}} \sum_{v_1=0}^{m-n} \cdots \sum_{v_{n-1}=0}^{m-n} \frac{\mathcal{T}(\mathbf{\Phi}_v) \prod_{i=1}^{n-1} \binom{m-n}{v_i} (x-n)^{B(v)-2}}{\Gamma(B(v)-1)}, \text{ for } x \geq n, \quad (6)$$

where $\Gamma(\cdot)$ denotes the gamma function, $K_{n,m}^{-1} = \prod_{i=1}^{n} \Gamma(m-i+1)\Gamma(n-i+1)$, $\boldsymbol{v} = [v_1 \ v_2 \ \cdots v_{n-1}]$, $B(\boldsymbol{v}) = n^2 + \sum_{i=1}^{n-1} v_i$, and $\mathbf{\Phi}_v = \{\Gamma(i+j+v_k+1)\}_{i,j,k=1,\ldots,n-1}$.

Figure 3: For rectangular $\mathbf{X} \sim \mathcal{CN}_{m,n}(\mathbf{0}, \mathbf{I}_m \otimes \mathbf{I}_n)$: n dependent sum and *tensor* $\mathcal{T}$

What We Know

"Distribution of Demmel and related condition numbers"
P. Dharmawansa et. al.
SIAM J. Matrix. Anal. Appl., 2013.

THEOREM 3.3. *The exact p.d.f. of $\kappa_D^2(\mathbf{A})$ is given by*

$$
\begin{aligned}
f^{(\alpha)}_{\kappa_D^2(\mathbf{A})}(y) &= \Gamma(mn)\left(\prod_{k=0}^{\alpha}\frac{n+k}{(k+1)!}\right)(y-n)^{mn-\alpha-2}y^{-mn}\\
&\times\sum_{j_1=0}^{n+\alpha-2}\cdots\sum_{j_\alpha=0}^{n-1}\left(\prod_{k=1}^{\alpha}(-1)^{j_k}\frac{(-n-\alpha+k+1)_{j_k}}{(k+2)_{j_k}\,j_k!}(y-n)^{-j_k}\right)\\
&\times\frac{\Delta_\alpha(\mathbf{c})}{\Gamma\left(mn-\alpha-1-\sum_{k=1}^{\alpha}j_k\right)}H(y-n),
\end{aligned}
\tag{3.9}
$$

where $\mathbf{c}=\{c_1(j_1),c_2(j_2),\ldots,c_\alpha(j_\alpha)\}$ *with* $c_l(j_l)=l+j_l$, *and* $H(z)$ *denotes the Heaviside unit step function, i.e.,* $H(z)=1,\ z\geq 0$, *and* $H(z)=0,\ z<0$.

Figure 4: For rectangular $\mathbf{X}\sim\mathcal{CN}_{m,n}(\mathbf{0},\mathbf{I}_m\otimes\mathbf{I}_n)$: $(m-n)$ dependent sum and det

What We Know

"Some new results on the eigenvalues of complex non-central Wishart matrices with rank-1 mean"
P. Dharmawansa
J. Multivariate. Anal., 2016.

Theorem 6. *Let* $\mathbf{W} \sim \mathcal{W}_n\left(m, \mathbf{I}_n, \mathbf{M}^*\mathbf{M}\right)$, *where* $\mathbf{M}$ *is rank-1 and* $\mathrm{tr}(\mathbf{M}^*\mathbf{M}) = \mu$. *Then the p.d.f. of* V *is given by*

$$f_V^{(\alpha)}(v) = (n-1)! \frac{e^{-\mu}}{v^{n(n+\alpha)}} \mathcal{L}^{-1}\left\{ \frac{e^{-ns}}{s^{(n-1)(n+\alpha+1)}} R(s, v, \mu) \right\} \tag{21}$$

where

$$R(s, v, \mu) = \det\left[\left(-\frac{\mu}{sv}\right)^{i-1} \phi_i(\mu, s, v)\, L^{(j)}_{n+i-1-j}(-s) \right]_{\substack{i=1,\ldots,\alpha+1 \\ j=2,\ldots,\alpha+1}}$$

$$\phi_i(\mu, s, v) = \sum_{k=0}^{\infty} \frac{a_i(k)}{k!} \left(\frac{\mu}{sv}\right)^k {}_1F_1\left(n^2 + n\alpha + k + i - 1; n + i + k + \alpha - 1; \frac{\mu}{v}\right)$$

$$a_i(k) = (n+i-1)\frac{(n^2 + n\alpha + i - 2)!}{(n+i+\alpha-2)!} \frac{(n+i)_k (n+i-2)_k (n^2 + n\alpha + i - 1)_k}{(n+i-1)_k (n+i+\alpha-1)_k}$$

and $\mathcal{L}^{-1}(\cdot)$ *denotes the inverse Laplace transform.*

Figure 5: For rectangular $\mathbf{X} \sim \mathcal{CN}_{m,n}(\boldsymbol{\mu}, \mathbf{I}_m \otimes \mathbf{I}_n)$: Support *smoothed analysis*

Extension: Correlated $\mathbf{X}$

The Journey

- $f\left(\mathbf{W}\right)\mathrm{d}\mathbf{W} = \dfrac{\det^{m-n}\left(\mathbf{W}\right)e^{-\mathrm{tr}\left(\mathbf{\Sigma}^{-1}\mathbf{W}\right)}}{\tilde{\Gamma}_n(m)\det^m\left(\mathbf{\Sigma}\right)}\mathrm{d}\mathbf{W}$ (Wishart, 1928)
- Eigen-decomposition $\mathbf{W} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^* \implies f\left(\mathbf{W}\right)\mathrm{d}\mathbf{W} \propto f\left(\mathbf{U}\mathbf{\Lambda}\mathbf{U}^*\right)\Delta^2(\boldsymbol{\lambda})\mathrm{d}\mathbf{\Lambda}\mathrm{d}\mathbf{U}$
  Eigenvectors + Eigenvalues
- $f(\lambda_1, .., \lambda_n) \propto \det^{-m}\left(\mathbf{\Sigma}\right)\Delta^2(\boldsymbol{\lambda})\prod_{j=1}^n \lambda_j^{m-n}\int_{\mathcal{U}_n} e^{-\mathrm{tr}\left(\mathbf{\Sigma}^{-1}\mathbf{U}\mathbf{\Lambda}\mathbf{U}^*\right)}\mathrm{d}\mathbf{U}$
- Integration over the unitary manifold: Contour integral approach (Wang, 2012)
- Joint eigenvalue density:

Vandermonde det.

$\Delta_n(\boldsymbol{\lambda}) = \prod_{1\le i<j\le n}(\lambda_j - \lambda_i)$

$$f(\lambda_1, .., \lambda_n) = C_{n,\alpha,\eta}\prod_{i=1}^n \lambda_i^\alpha e^{-\lambda_i}\,\Delta_n^2(\boldsymbol{\lambda})\sum_{k=1}^n \frac{e^{c_\eta \lambda_k}}{\prod_{\substack{i=1\\ i\neq k}}^n (\lambda_k - \lambda_i)}$$

The Journey

- Derive m.g.f.:
$\mathcal{M}_{\kappa^2_{\mathsf{SC}}(\mathbf{X})}(s) = \mathbb{E}\left\{e^{-s\kappa^2_{\mathsf{SC}}(\mathbf{X})}\right\} = \int_{0<\lambda_1\le\lambda_2\le\cdots\le\lambda_n<\infty} e^{-s\kappa^2_{\mathsf{SC}}(\mathbf{X})} f(\lambda_1,\dots,\lambda_n)\mathrm{d}\lambda_1\dots\mathrm{d}\lambda_n$
- Selberg-type integrals on $\mathbb{R}^n$

$$T_n^{(\alpha)}(a,b) := \underbrace{\int\cdots\int}_{[0,\infty)^n} \prod_{i=1}^{n}(a-y_i)(b-y_i)^{\alpha} \underbrace{e^{-y_i}y_i^2}_{\text{Laguerre weight}} \Delta_n^2(\mathbf{y})\mathrm{d}y_1\mathrm{d}y_2\cdots\mathrm{d}y_n$$

Solution: Orthogonal polynomial technique (Mehta, 2004)

- Laplace inverse: $f^{\alpha}_{\kappa^2_{\mathsf{SC}}(\mathbf{X})}(z) = \mathcal{L}^{-1}\left\{\mathcal{M}_{\kappa^2_{\mathsf{SC}}(\mathbf{X})}(s)\right\}$

Finite Dimensional Result

Theorem: The p.d.f. of $\kappa^2_{\mathsf{SC}}(\mathbf{X})$, for $\mathbf{X} \sim \mathcal{CN}_{n,m}(\mathbf{0}, (\mathbf{I}_n + \eta \mathbf{u}\mathbf{u}^*) \otimes \mathbf{I}_m)$, is given by

$$f^{\alpha}_{\kappa^2_{\mathsf{SC}}(\mathbf{X})}(z) = K_{n,\alpha,\eta} \frac{(z-n)^{n(n+\alpha)-\alpha-2}}{\left(z - \frac{\eta}{1+\eta}\right)^{n(n+\alpha)}} \boxed{\sum_{k_1=0}^{n+\alpha-2} \cdots \sum_{k_\alpha=0}^{n-1}} \prod_{j=1}^{\alpha} A_j(k_j)(z-n)^{-k_j}$$

Step function

$$\times \det \left[\mathcal{Q}_i(z,\eta) \quad \frac{1}{\Gamma(n+i-j-k_j)} \right]_{\boxed{\substack{i=0,..,\alpha \\ j=1,..,\alpha}}} \boxed{H(z-n)}$$

Depend only on α

$\mathcal{Q}_i(z,\eta) \in \mathbb{R}$ is in terms of ${}_3F_2\,(\ldots)$.

- Complexity depends on $\alpha = m - n$
- Facilitates asymptotic analysis (for large m, n)
- For $\alpha = 0$: *simple expression*

$f^0_{\kappa^2(\mathbf{X})}(z) \propto \frac{(z-n)^{n^2-2}}{(z-c_\eta)^{n^2}} \; {}_3F_2 \left(\text{arguments in } n; c_\eta \frac{z-n}{z-c_\eta} \right) H(z-n)$

Analysis vs. Intuition

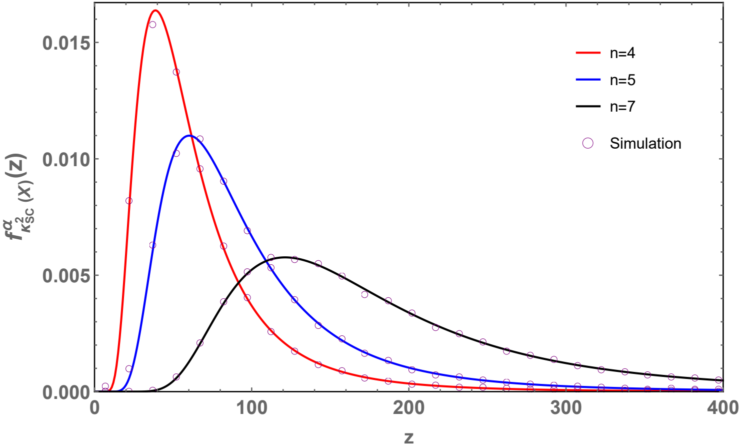

Figure 6: Effect of matrix size (m, n)

Analysis vs. Intuition

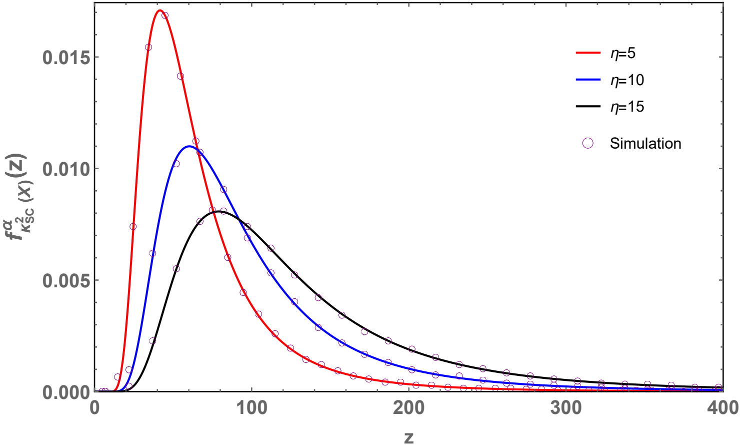

Figure 7: Effect of correlation (η)

Analysis vs. Intuition

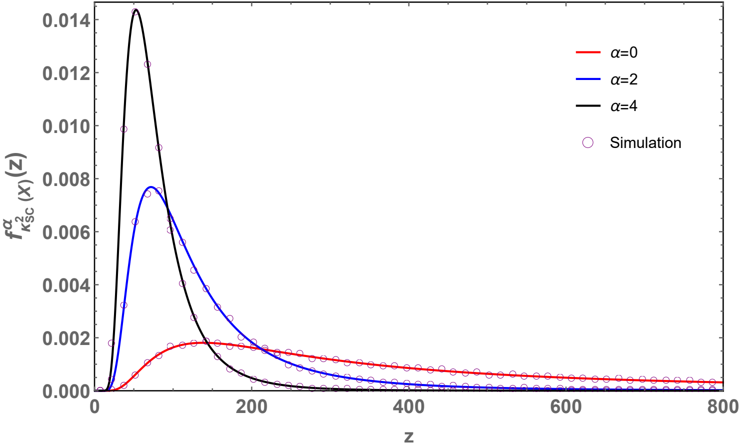

Figure 8: Effect of m with fixed n

- Large m with n, η fixed $\implies$ p.d.f. concentrates around $n + \eta$

Detector Performance: ROC

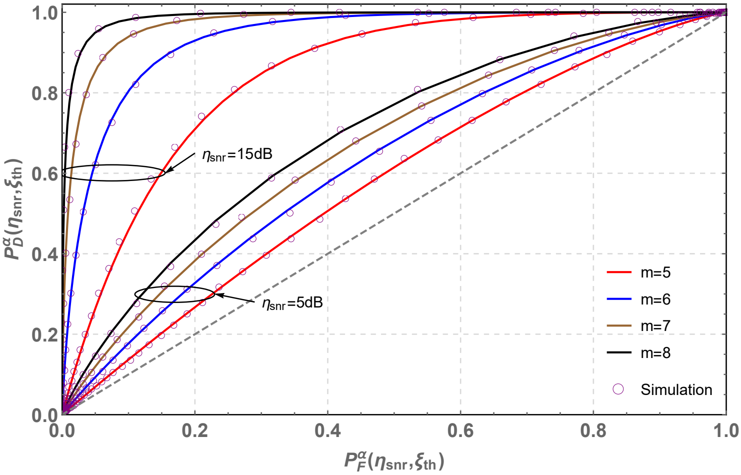

Figure 9: Effect of m and η_{SNR} with $n = 5$

$P_F^{\alpha} = \Pr\left\{\kappa_{\mathsf{SC}}^2(\mathbf{X}) > \xi | \mathcal{H}_0\right\}$ $\quad P_D^{\alpha} = \Pr\left\{\kappa_{\mathsf{SC}}^2(\mathbf{X}) > \xi | \mathcal{H}_1\right\}$

Asymptotic Behaviour

Limit $m, n \to \infty$ with $n/m = c \in (0, 1)$ constant

- Scaling: $\eta = O(1)$ and $\dfrac{(1-\sqrt{c})^{8/3}}{c^{5/6} m^{1/3}} \left(\kappa_{\mathsf{SC}}^2(\mathbf{X}) - \dfrac{mc}{(1-\sqrt{c})^2} \right) = W$

$$F_W^c(w) \to F_2(w)$$

- $F_2(w)$ is the famous Tracy-Widom distribution (Tracy and Widom, 1994)

$$F_2(t) = \exp\left(-\int_t^\infty (x-t) q^2(x) \mathrm{d}x \right)$$

$q(x)$ denotes the Hastings-McLeod solution of the homogeneous Painlevé II equation $\frac{\mathrm{d}^2}{\mathrm{d}x^2} q(x) = 2q^3(x) + xq(x)$ characterized by the boundary condition $q(x) \sim \mathrm{Ai}(x)$ as $x \to \infty$ with $\mathrm{Ai}(x)$ denoting the Airy function

Asymptotic Behaviour

Limit $m, n \to \infty$ with $m - n = \alpha$ constant (*Phy. macroscopic limit*)

- Scaling: $\eta \propto 1/n$ and $\kappa^2_{\mathsf{SC}}(\mathbf{A})/\mu n^3 = V_n \xrightarrow{\text{in distribution}} V$

$$F_V^{\alpha}(v) = e^{-\frac{1}{\mu v}} \det\left[I_{j-i}\left(\frac{2}{\sqrt{\mu v}}\right)\right]_{i,j=1,..,\alpha} H(v)$$

- For $\alpha = 0$, $F_V^0(v) = e^{-\frac{1}{\mu v}}$
- No η-dependency $\implies$ Coincides: uncorrelated $\mathbf{X}$ (Dharmawansa *et. al.*, 2013; Dharmawansa, 2016)

How Small is too Small?

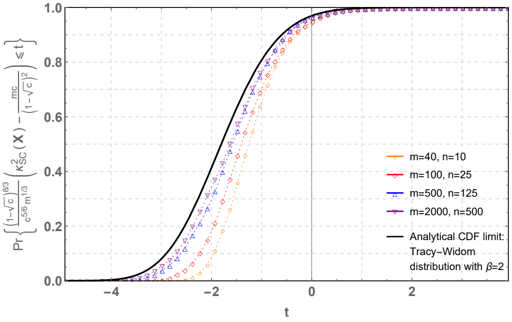

Figure 10: Asymptotic c.d.f.: constant n/m

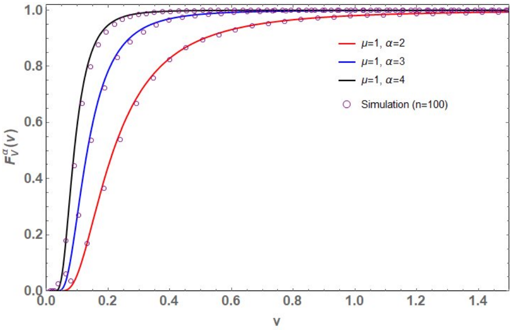

Figure 11: Asymptotic c.d.f.: constant $m - n$ (simulations with $n = 100$)

- Agrees well for moderately large finite dimensions

Conclusion

- Correlated Wishart matrix: Single-spiked covariance $\mathbf{\Sigma} = \mathbf{I}_n + \eta \mathbf{u}\mathbf{u}^*$
- Scaled condition number

$$\kappa_{\mathsf{SC}}(\mathbf{X}) = \sqrt{\frac{\sum_{i=1}^{n} \lambda_i}{\lambda_1}}$$

- Exact p.d.f.: Complexity depends on rectangularity of $\mathbf{X}$ (i.e., $m - n$)
- Asymptotic characterization in two regimes
- Extension to *rank-r perturbation*: $\mathbf{\Sigma} = \mathbf{I}_n + \sum_{k=1}^{r} \eta_k \mathbf{u}_k \mathbf{u}_k^*$??
- A technical paper to appear in IEEE Transactions on Information Theory

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Thank You! 😄